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CONNECTIVITY OF CIRCULANT DIGRAPHS

Preprint

Connectivity of circulant digraphs[*)]

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ABSTRACT

An explicit expression is derived for the connectivity of circulant digraphs.

KEY WORDS & PHRASES: *circulant (digraph), connectivity*

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1. Introduction

A <u>digraph</u> D consists of a finite set of <u>vertices</u> V(D) and a set of <u>arcs</u> A(D) which is a subset of all ordered pairs (u,v), u,v $\in$ V(D), and u $\neq$ v. If (u,v) $\in$ A(D), then u is a <u>predecessor</u> of v and v is a <u>successor</u> of u.

For a subset U $\subset$ V(D), D - U denotes the subgraph of D induced by V(D) - U. Furthermore, the set

$$0_D(U) = \{v \in V(D) - U : (u,v) \in A(D) \text{ for some } u \in U\}$$

is called the <u>outset</u> of U in D. If U = {u} we write u instead of {u} in the above expressions. Also, if E is a subgraph of D we write $0_D(E)$ instead of $0_D(V(E))$.

A <u>cut set</u> C of D is a proper subset of V(D) such that D - C either is not strongly connected or consists of a single vertex. C is a <u>minimum cut set</u> if any other cut set of D has at least |C| vertices. The size of a minimum cut set is denoted by $\kappa(D)$ and called the <u>connectivity</u> of D. Clearly, $\kappa(D) < |V(D)| - 1$ unless D is complete.

For a fixed integer n > 0 and set S $\subset$ {1,2,...,n-1} we construct a digraph D = D(n,S) with vertices $v_0, v_1, \ldots, v_{n-1}$ as follows: $(v_i, v_j) \in A(D)$ if and only if j - i $\equiv$ s (mod n) for some s $\in$ S. Any digraph that can be constructed in the above manner is called a <u>circulant digraph</u> (or <u>directed star polygon</u>). In this note we derive an explicit expression for the connectivity $\kappa(D)$ of a circulant digraph D = D(n,S) in terms of its order n and its <u>symbol</u> S.

Evidently, for any vertex $v \in V(D)$ of a circulant digraph $D = D(n,S)$ one has

$$|\mathcal{O}_D(v)| = |S| , \qquad (1)$$

so that an upperbound for the connectivity of D is given by

$$\kappa(D) \leq |S| , \qquad (2)$$

since removal of $\mathcal{O}_D(v)$ from D disrupts all directed paths from v to any other vertex in $D - \mathcal{O}_D(v)$. If equality holds in (2) the circulant digraph D is said to be κ-optimal. It is of interest in network design studies to be able to identify the class of circulant digraphs which are κ-optimal. Our basic result on the connectivity of circulant digraphs, which is stated in Section 3, enables us to perform this identification. It will subsequently be shown that the known sufficient conditions for κ-optimality of circulant digraphs follow readily from our result. Section 2 contains some preliminary results.

If the symbol of a circulant digraph $D = D(n,S)$ is such that $s \in S \iff n-s \in S$, so that the arc $(v_i, v_j) \in A(D)$ if and only if $(v_j, v_i) \in A(D)$, we can replace $D(n,S)$ by the corresponding graph $G(n,S)$ in which v_i is adjacent to v_j if and only if the pair of arcs (v_i, v_j) and (v_j, v_i) are in $A(D)$. Any graph that can be obtained in this way is a circulant graph. Defining connectivity for graphs in the usual way [6], a circulant graph and its corresponding di-

graph have the same connectivity. The recently obtained necessary and sufficient condition for κ-optimality of circulant graphs [4] and all previously known sufficient conditions (see the references in [4]) can therefore be obtained from our result by thinking of such a graph as the corresponding digraph.

In what follows circulant stands for circulant digraph. For concepts used but not defined here we refer to Harary's book [6].

2. Modules and atomic traps

We consider a circulant $D = D(n,S)$ with vertices $v_0, v_1, \ldots, v_{n-1}$. Let r be a divisor of n (notation: $r|n$). For $i = 0,1,\ldots,r-1$ we denote by $F_r^i = F_r^i(D)$ the subgraph of D which is induced by the set of vertices $V(F_r^i) = \{v_i, v_{i+r}, v_{i+2r}, \ldots, v_{i+n-r}\}$. Any such subgraph F_r^i will be called a __module__ of D (of order n/r). We next define

$$M(r,S) = \{\ell : 1 \le \ell < r \text{ and } \ell \equiv s \pmod{r} \text{ for some } s \in S\}, \quad (3)$$

so that, in particular, $M(n,S) = S$ and $M(1,S) = \emptyset$.

LEMMA 1. For any $r|n$ and $i \in \{0,1,\ldots,r-1\}$ one has

$$|\mathcal{O}_D(F_r^i)| = \frac{n}{r}|M(r,S)| . \quad (4)$$

Proof. It is clear that if $V(F_r^j)$ contains a successor of a vertex in $V(F_r^i)$, $i \ne j$, then each vertex of $V(F_r^j)$ is a successor of a vertex in $V(F_r^i)$. Hence, $|\mathcal{O}_D(F_r^i)|$ is n/r times the number of modules F_r^j, $j \ne i$, which contain successors of a vertex in F_r^i. But then, one has $(v_{i+kr}, v_{j+\ell r}) \in A(D)$ for some k and ℓ ($0 \le k, \ell < n/r$) if and only if either $j > i$ and $j-i \in M(r,S)$ or $j < i$ and $r+j-i \in M(r,S)$, whence this number of modules equals $|M(r,S)|$. $\square$

If $|\mathcal{O}_D(F_r^i)| < n - n/r$, or, equivalently, $|M(r,S)| < r - 1$, then $\mathcal{O}_D(F_r^i)$ constitutes a cut set. We can therefore state the following.

LEMMA 2. For each $r|n$ such that $|M(r,S)| < r - 1$ one has

$$\kappa(D) \leq \frac{n}{r}|M(r,S)| \, . \tag{5}$$

In what follows we will show that equality holds in (5) for some $r|n$. The main obstacle in establishing this result is a proof for the fact that D contains a minimum cut set which is the outset of a module of D. The remainder of this section will be dedicated to this task. We start off with some terminology.

The strongly connected components of the subgraph $D - C$ of a digraph D, where C is a minimum cut set of D, are called parts of D with respect to C. A part T of D with respect to C whose vertices have no successors outside T in the subgraph $D - C$ will be called a trap of D with respect to C. Clearly, for each minimum cut set of D there is at least one trap. Furthermore, it is readily verified that if T is a trap of D with respect to the minimum cut set C, then C consists precisely of the successors of the vertices in T which are outside T, that is, $C = 0_D(T)$. A trap T of D with the property that $|V(T)|$ does not exceed the order of any other trap of D is an atomic trap. The order of an atomic trap will be denoted by $a(D)$.

We will investigate the structure of an atomic trap of a circulant. It will be convenient, however, to first consider self-converse digraphs, that is, digraphs $D = (V(D),A(D))$ with the property that D is isomorphic to its converse $D' = (V(D'),A(D'))$, where $V(D') = V(D)$ and $A(D') = \{(u,v) : (v,u) \in A(D)\}$. The next lemma,

which is easy to verify (cf. the proof of Proposition 3 in [1]),

shows that this context is more general.

LEMMA 3. A circulant is self-converse.

Concerning self-converse digraphs we have the following results.

LEMMA 4. Let D be a self-converse digraph which is not complete.

If T is an atomic trap of D with respect to C = 0_D(T), then

$|V(D - (C \cup V(T)))| \geq a(D)$.

Proof. Since D is not complete, the subgraph D - C contains at

least two parts with respect to C of which T is one. Clearly, there

is at least one part P $\neq$ T whose vertices have no predecessors out-

side P in D - C. P is a trap with respect to C in the converse of

D, and since D is self-converse it follows that $|V(P)| \geq a(D)$. Hence,

$|V(D - (C \cup V(T)))| \geq |V(P)| \geq a(D)$. $\square$

LEMMA 5. In a self-converse digraph distinct atomic traps are

disjoint.

Proof. The lemma is obviously true if D is complete. Therefore,

assume that D is not complete.

Let A and B be two atomic traps of D with vertex sets V_A and V_B

and corresponding cut sets $C_A = 0_D$(A) and $C_B = 0_D$(B), respectively.

Thus $|V_A| = |V_B| = a(D)$ and $|C_A| = |C_B| = \kappa(D)$. Also, by Lemma 4,

$$|V_D - (C_B \cup V_B)| \geq a(D) > 0 , \qquad (6)$$

where $V_D = V(D)$. Assuming that $V_A \cap V_B \neq \emptyset$, we let

$$S_1 = 0_D(V_A \cap V_B) \quad \text{and} \quad S_2 = 0_D(V_A \cup V_B) .$$

Evidently, $S_1 \cup (V_A \cap V_B) \subset C_B \cup V_B$, so that, by (6),

$$|V_D - (S_1 \cup (V_A \cap V_B))| \geq |V_D - (C_B \cup V_B)| > 0 .$$

It follows that S_1 is a cut set. Hence, since

$$S_1 \subset ((C_A \cup V_A) \cap (C_B \cup V_B)) - (V_A \cap V_B) =$$

$$= (C_A \cap V_B) \cup (C_B \cap V_A) \cup (C_A \cap C_B) ,$$

we have

$$|C_A \cap V_B| + |C_A - V_B| = |C_A| = \kappa(D) \leq |S_1| \leq$$

$$\leq |C_A \cap V_B| + |C_B \cap V_A| + |C_A \cap C_B| , \qquad (7)$$

so that

$$|C_A - V_B| \leq |C_B \cap V_A| + |C_A \cap C_B| . \qquad (8)$$

We subsequently note that, by (8),

$$|(C_A \cup V_A) - (C_B \cup V_B)| =$$

$$= |(V_A - V_B - (C_B \cap V_A)) \cup (C_A - V_B - (C_A \cap C_B))| =$$

$$= |V_A - V_B| - |C_B \cap V_A| + |C_A - V_B| - |C_A \cap C_B| \leq$$

$$\leq |V_A - V_B| < |V_A| = a(D) . \tag{9}$$

Since $S_2 \cup (V_A \cup V_B) = C_A \cup C_B \cup V_A \cup V_B$, it follows by (6) and (9) that

$$|V_D - (S_2 \cup (V_A \cup V_B))| =$$

$$= |V_D - (C_B \cup V_B)| - |(C_A \cup V_A) - (C_B \cup V_B)| > 0 ,$$

so that S_2 is a cut set too. Hence, since

$$S_2 = (C_A \cup C_B) - (V_A \cup V_B) =$$

$$= (C_A - V_B) \cup (C_B - V_A - (C_A \cap C_B)) ,$$

we have, using (8),

$$\kappa(D) \leq |S_2| = |C_A - V_B| + |C_B - V_A| - |C_A \cap C_B| \leq$$

$$\leq |C_B - V_A| + |C_B \cap V_A| = |C_B| = \kappa(D) ,$$

so that equality holds in (8). Using this result in (7) we obtain $|S_1| = \kappa(D)$, i.e., S_1 is a minimum cut set. It follows that the subgraph of D induced by $V_A \cap V_B$ is a trap of D, and since A and B are atomic traps we must have A = B. $\square$

REMARK. Lemma 5 is an extension of Watkins' fundamental Theorem 1
on graphs [9]. In our proof we have used some of the ideas of
Tindell's alternative (but incorrect) proof for Watkins' result [8].

We are now ready to state the key result of this section.

LEMMA 6. An atomic trap of a circulant D is a module of D of order
a(D).

Proof. Let $V(D) = \{v_0, v_1, \ldots, v_{n-1}\}$ be the vertex set of the cir-
culant $D = D(n,S)$, and let T be an atomic trap of D with vertex set
$V(T) = \{v_{t_1}, v_{t_2}, \ldots, v_{t_a}\}$, where $a = a(D)$. If $a = 1$ the lemma is
trivially true, therefore assume that $a > 1$. Because of the invari-
ance of the connection relations under cyclic permutations of the
vertex indices it is no restriction to assume that $t_1 = 0$ and

$$|0_T(v_0)| = \max \{|0_T(v_i)| : v_i \in V(T)\} . \qquad (10)$$

The set of indices $\{t_1=0, t_2, \ldots, t_a\}$ of the vertices in T will be
denoted by $I(T)$. It will be convenient to interprete vertex indices
as elements in the ring $\mathbb{Z}_n$ of integers modulo n.

Since an atomic trap is strongly connected, we have
$0_D(v_0) \cap V(T) \neq \emptyset$, whence $S \cap I(T) \neq \emptyset$. Obviously, the subgraph of
D which is induced by the vertices with indices in $I(T) + s =$
$\{s, t_2+s, \ldots, t_a+s\}$, where $s \in S \cap I(T)$, is also an atomic trap.
Since $I(T)$ and $I(T) + s$ have a common element s, it follows by

Lemma 5 that $I(T) = I(T) + s$. This result being valid for each $s \in S \cap I(T)$, we conclude that $I(T)$ contains the elements of the ideal of $\mathbb{Z}_n$ which is produced by the elements of $S \cap I(T)$. Let E denote the subgraph of D which is induced by the vertices whose indices are in this ideal. Clearly, E is a module of D and we are done if we can show that $V(T) = V(E)$.

To this end we subsequently note that $O_T(v_0) \subset V(E)$, whence $|O_E(v_0)| = |O_T(v_0)|$. Moreover, for each $v \in V(E)$ one clearly has $|O_E(v)| = |O_E(v_0)|$. In view of (10) we have $|O_E(v)| \leq |O_T(v)| \leq |O_T(v_0)|$, so that, actually, $|O_E(v)| = |O_T(v)|$ for each $v \in V(E)$. Since T is strongly connected it follows that $V(E) = V(T)$. $\square$

3. Connectivity of circulants

By Lemma 6 an atomic trap of the circulant $D = D(n,S)$ is a module of D of order $a = a(D)$. Since the minimum cut set corresponding to an atomic trap is the outset of that atomic trap, it follows by Lemma 1 that

$$\kappa(D) = a|M(\frac{n}{a},S)| \; .\qquad\qquad(11)$$

If D is complete, then $a = 1$ and $\kappa(D) = n - 1$. On the other hand, if D is not complete, then removal of the minimum cut set corresponding to an atomic trap leaves at least two parts, whence $\kappa(D) < n - a$, i.e., $|M(\frac{n}{a},S)| < \frac{n}{a} - 1$. Lemma 2 subsequently yields our main result.

THEOREM 1. The connectivity $\kappa(D)$ of a circulant $D = D(n,S)$ which is not complete is given by

$$\kappa(D) = \min \; \{\frac{n}{r}|M(r,S)| \; : \; r|n \text{ and } |M(r,S)| < r - 1\} \; . \quad (12)$$

EXAMPLE. The circulant $D(12,\{1,4,5,9\})$ has connectivity 3 as appears from the results of Table 1.

Some corollaries to Theorem 1 will now be derived. First we will give a condition for a circulant D to be strongly connected, i.e., $\kappa(D) \geq 1$, which was mentioned earlier in [2].

r	M(r,S)	$\|M(r,S)\| < r-1$?	$\frac{n}{r}\|M(r,S)\|$
1	$\emptyset$	no	(0)
2	{1}	no	(6)
3	{1,2}	no	(8)
4	{1 }	yes	3
6	{1,3,4,5}	yes	8
12	{1,4,5,9}	yes	4

Table 1. Connectivity analysis for the circulant of

order 12 with symbol $S = \{1,4,5,9\}$.

COROLLARY 1.   The circulant $D(n,S)$, where $S = \{s_1, s_2, \ldots, s_{|S|}\}$ is

strongly connected if and only if $g.c.d.(n, s_1, s_2, \ldots, s_{|S|}) = 1$.

Proof.   By Theorem 1 we have $\kappa(D) \geq 1$ if and only if $|M(r,S)| > 0$

for all $r|n$, $r > 1$. The statement follows readily. $\square$

The most important problem pertaining to connectivity of cir-

culants is to determine whether a given circulant $D(n,S)$ is

κ-optimal. A simple sufficient condition is given in the next cor-

ollary.

COROLLARY 2.   The circulant $D(n,S)$ is κ-optimal if $g.c.d.(n,s) = 1$

for all $s \in S$.

Proof. Let T be an atomic trap of $D = D(n,S)$ and assume $v_0 \in V(T)$.

If $|V(T)| = a(D) = 1$, then, by (11), $\kappa(D) = |M(n,S)| = |S|$, so that D is κ-optimal. Next suppose $|V(T)| > 1$. T is strongly connected, whence there exists an $s \in S$ such that $v_s \in V(T)$. Since g.c.d.$(n,s) = 1$ and T is a module of D, it follows that T = D, which is a contradiction. □

In particular, D(n,S) is κ-optimal if n is prime.

The next sufficient condition for κ-optimality was stated (without proof) in [5] and (with an incorrect proof) in [7].

COROLLARY 3*. If the circulant D(n,S) is such that $i \in S$ for $i = 1,2,\ldots,\lceil |S|/2 \rceil$, then D(n,S) is κ-optimal.

Proof. If $r|n$ and $n > r > \lceil |S|/2 \rceil$, then $|M(r,S)| \geq |S|/2$. Since $n/r \geq 2$, it follows that $\frac{n}{r}|M(r,S)| \geq |S|$. If, on the other hand, $r \leq \lceil |S|/2 \rceil$, then $|M(r,S)| = r - 1$. By Theorem 1 we therefore have $\kappa(D) = |M(n,S)| = |S|$. □

We finally remark that κ-optimality for the more special case where $S = \{1,2,\ldots,|S|\}$ was proven in [3].

* $\lceil x \rceil$ denotes the smallest integer greater than or equal to x.

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